Biased Random Walk

One route to make Gaussian f.p. unstable is to break the rotational symmetry of the p.d.f. Consider

$$p(\vec{r}) = \frac{e^{-\frac{(\vec{r}-\vec{r}_0)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}.$$

One now anticipates that each step will approximately correspond to a displacement by $\vec{r}_0$ and therefore, the displacement after n steps will be proportional to n (unlike the Gaussian f.p., where it is $\propto \sqrt{n}$). Therefore we expect that the Gaussian f.p. will be unstable to such a bias. To confirm this, lets do a simple RG calculation.

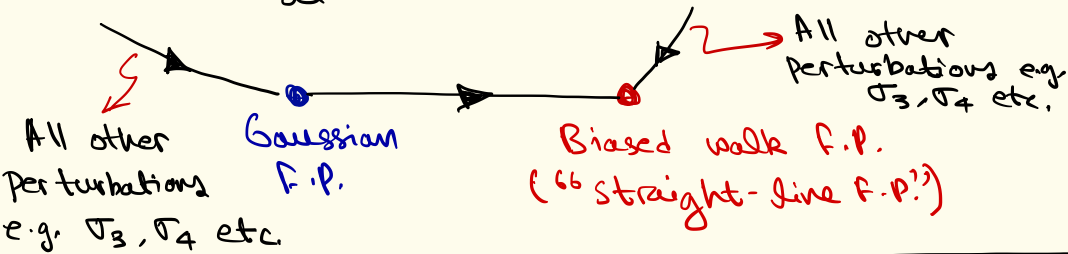
Fourier transforming, $p(\vec{k}) \sim e^{i\vec{k}\cdot\vec{r}_0 - \frac{k^2\sigma^2}{2}}$

Coarse-graining (for $r' = r_1 + r_2$) : $p(\vec{k}) \to e^{2i\vec{k}\cdot\vec{r}_0 - k^2\sigma^2}$

Re-scaling : lets again perturb at the Gaussian f.p. so that $(k'')^2 = 2k^2$.

$$\Rightarrow \quad p(\vec{k}) \to e^{\sqrt{2}\, i\vec{k}''\cdot\vec{r}_0 - \frac{(k'')^2\sigma^2}{2}}$$

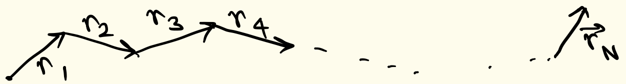
Therefore, the bias term increases under RG. : $\frac{d|\vec{r}_0|}{dl} = |\vec{r}_0| \log(\sqrt{2})$. The RG flow is:



Self-Avoiding Walk (SAW): a stable non-Gaussian fixed-point

The above discussion may give the impression that in the presence of rotational sym. of the pdf corresponding to each step of the walk, the Gaussian F.P. is the only stable one. It turns out this is conclusion is incorrect. One way to obtain a stable non-Gaussian F.P. is to consider <u>walks with memory</u>. The prime example is the <u>self-avoiding walk</u> which is a walk that is not allowed to intersect itself.

To gain some intuition about self-avoiding walk, it will be useful to formulate a continuum description. Consider a walk made of vectors $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N$:



Denoting the global position vector corresponding to the end of vector $\vec{r}_n$ as $\vec{R}_n$, one obtains $\vec{r}_n = \vec{R}_n - \vec{R}_{n-1}$. The probability distribution for a Gaussian random walk (i.e. allowing self-intersections) is :

$$p \propto e^{-\sum_n \frac{r_n^2}{2\sigma^2}}$$

$$= e^{-(\vec{R}_n - \vec{R}_{n-1})^2/2\sigma^2}$$

$$\approx e^{-\int \frac{1}{2\sigma^2}\left(\frac{\partial \vec{R}}{\partial s}\right)^2 ds}$$

where 's' parametrizes the distance along the walk.

To penalize self-intersections, one now simply adds the following term to the p.d.f. :

$$u \int ds_1\, ds_2\, \delta^d(\vec{R}(s_1) - \vec{R}(s_2))$$

so that

$$p \propto e^{-\left[\int dt \frac{1}{2\sigma^2}\left(\frac{\partial R}{\partial s}\right)^2 + u\int ds_1\, ds_2\ \delta^d(R(s_1)-R(s_2))\right]}$$

The penalty term energetically disfavors self-intersections, thus modeling the physics of a self-avoiding walk (SAW).

Two basic questions are :

1) Does the 'u' term alters the long-distance properties of the Gaussian random walk ? e.g. do we expect that the average displacement will no longer scale with $\sqrt{N}$ where $N = \int dt$ is the size of the walk.

2) If the answer to #1 is 'Yes', can one develop an RG framework to predict universal properties of a SAW.

Here we will restrict ourselves to only a very rudimentary treatment to these two questions.

To answer 1, let's change our notation a little bit so that the dimension of various variables is apparent. Let's write,

$$\frac{1}{\sigma^2}\int ds\left(\frac{\partial \vec{R}}{\partial s}\right)^2 = \int dt\left(\frac{\partial \vec{R}}{\partial t}\right)^2$$

where we have introduced $t \sim \sigma^2 s$ as a new variable, with units of $(\text{length})^2$. The interaction term becomes

$$u\,\sigma^{d-4}\int dt_1\, dt_2\, \delta^d[\vec{r}(t_1) - \vec{r}(t_2)]$$

where u is now dimensionless as one may check. At the Gaussian fixed point, $|\vec{R}| \sim \sqrt{t}$, which is consistent with the σ independence of the term $\int\left(\frac{\partial \vec{R}}{\partial t}\right)^2 dt$. Therefore, the interaction term vanishes in the continuum limit as $\sigma \to 0$ when $d > 4$, and blows up when $d < 4$. This suggests that the SAW has a different scaling behavior compared to the Gaussian walk for $d<4$ and same behavior as the Gaussian walk for $d>4$.

One may also justify the same conclusion using RG:

Coarse Graining: Define $t' = \frac{t}{b}$ $(b > 1)$. So that

$$-\log(p) =$$

$$\frac{1}{b}\int dt' \left[\frac{\partial \vec{R}(t'b)}{\partial t'}\right]^2 + \sigma^{d-4} u\, b^2 \int dt'_1\, dt'_2\, \delta^d[\vec{R}(t'_1 b) - \vec{R}(t'_2 b)]$$

Rescaling: Define $\vec{R}(t'b) = \sqrt{b}\, \vec{R}'(t')$, so that

$$-\log(p) = \int dt' \left[\frac{\partial \vec{R}'(t')}{\partial t'}\right]^2 + \sigma^{d-4} u\, b^{2-\frac{d}{2}} \int dt'_1 dt'_2\, \delta^d[\vec{R}'(t'_1) - \vec{R}'(t'_2)]$$

Thus, under RG at the Gaussian fixed point

$$u \longrightarrow u\, b^{2-\frac{d}{2}}$$

For $d < 4$, u grows under this RG procedure (recall $b > 1$), while for $d > 4$, it diminishes. Thus, one arrives at the same conclusion.

Universal Distinction between SAW and Gaussian Walk.

In the Gaussian walk, the averge root mean-square displacement $\sqrt{\langle R^2 \rangle} \sim \sqrt{N}$ after N steps. Since an SAW can't intersect itself, one might expect that for an SAW, $\sqrt{\langle R^2 \rangle} \sim N^{\nu}$ where $\nu > \frac{1}{2}$. This is indeed the case.

Here's a heuristic argument that provides an estimate of ν:

At the SAW fixed point, consider a rescaling of the 'cut-off' σ : $\sigma \to b\sigma$. Under this rescaling, lets assume $\vec{r} \to b^{-x}\vec{r}$ where x is some unknown. Note that at Gaussian fixed point, $x = 0$, since the Gaussian term $\int \left(\frac{\partial \vec{R}}{\partial t}\right)^2 dt$ is independent of σ.

Thus, under this rescaling the Gaussian term acquires a factor of b^{-2x} while the interaction term acquires a factor of b^{dx+d-4}.

At the SAW fixed point, one expects that both terms scale in the same way, so that there is a unique power-law scaling. If so,

$$-2x = d-4+dx$$

$$\Rightarrow \quad x = \frac{4-d}{d+2}$$

Writing $|\vec{R}| \sim \sigma f(t/\sigma^2)$, the scaling $|\vec{R}| \sim \frac{1}{b^x}$ under $\sigma \to b\sigma$ requires $f(y) \sim y^{(1+x)/2}$. Therefore $|\vec{R}| \sim t^{(1+x)/2}$

Setting $t = N$, where N is the # of steps,

$$|\vec{R}| \sim N^{(1+x)/2} = N^{\nu} \quad \text{where}$$

$$\nu = \frac{1+x}{2} = \frac{3}{d+2}.$$

As expected, in $d=1$, $\nu=1$ and in $d=4$, $\nu=\frac{1}{2}$ (Gaussian).

In $d=2$, $\nu=\frac{3}{4}$ turns out to be exact while for $d=3$ $\nu=\frac{3}{5}$ is incorrect (recall the above argument is heuristic).

RG for SAW: this is technically challenging. As we will later see that SAW can be mapped to an $O(N)$ model in $N \to 0$ limit. Therefore, any RG scheme for $O(N)$ model which allows for such an analytic continuation is also an RG for the SAW. For now we will just provide a heuristic real-space RG, similar to our discussion for the Gaussian walk.

Following Creswick, real space RG for SAW is easier to perform in a 'grand-canonical ensemble'.

First, lets define $M_n(\vec{R})$ = # of walks of n steps that connect origin to the point $\vec{R}$. One expects that M_n grows exponentially with n. The 'grand-canonical partition f^n' for a random walk is defined as

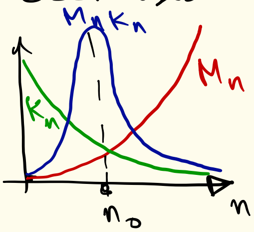
$$Z[K] = \sum_n K^n \sum_R M_n(\vec{R})$$

where K is a parameter (analog of $e^{\beta\mu}$ in partition f^n of particles). We will call it 'fugacity' in analog with a system consisting of particles.

One may now define a correlation length $\xi(K)$ as

$$\xi^2(K) = \frac{\sum_{\vec{R}} \sum_n |\vec{R}|^2 \, K^n M_n(\vec{R})}{Z(K)}$$

$\xi(K)$ corresponds to typical size of walk with fugacity K. Let's evaluate $\xi(K)$ within a saddle point approximation. $M_n(R)$ increases exponentially with n while for $K<1$, K^n decreases exponentially with n. Their product



is peaked at a specific $n = n_0$. When K is small, only walks with small n dominate. As K increases, n_0 diverges at a specific K_c. Writing $M_n = K_c^{-n} n^{\gamma-1}$ where γ is a universal number [homework: argue that $\gamma=1$ for a Gaussian walk],

$$K^n M_n = e^{n \log \frac{K}{K_c} + (\gamma-1)\log n}$$

At saddle pt.,
$$n_0(K) = \frac{\gamma-1}{\log(K_c/K)}$$

$$\simeq (\gamma-1)\left[\frac{K_c}{K_c - K}\right]$$

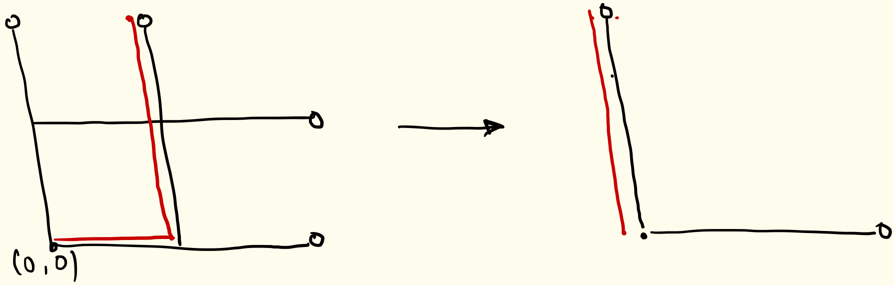
for $K \simeq K_c$.

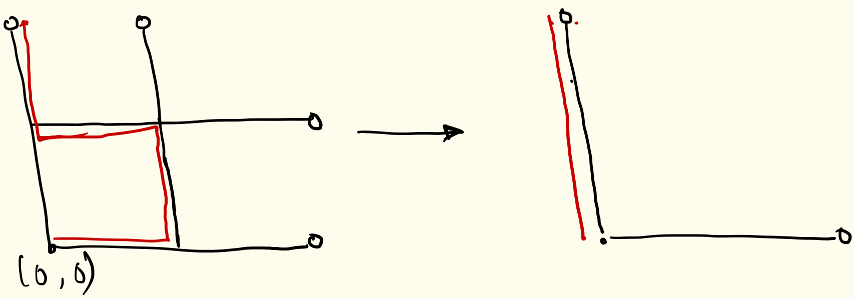
Using $R \sim N^{\nu}$, this implies that

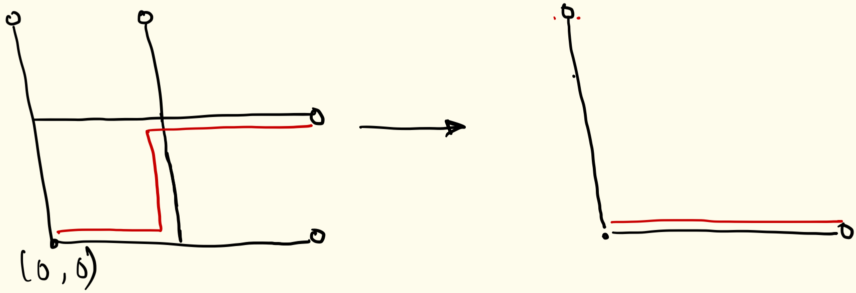
$$\xi(K) \sim n_0^{\nu} \sim \left[\frac{K_c - K}{K_c}\right]^{-\nu}$$

We are now ready to perform a real space RG. The key point is that under RG, the partition f^n must be invariant. We consider a RG procedure where a self-avoiding walk over a region of linear length 'b' is mapped to a self-avoiding walk of a single step. For example:

$b = 2$







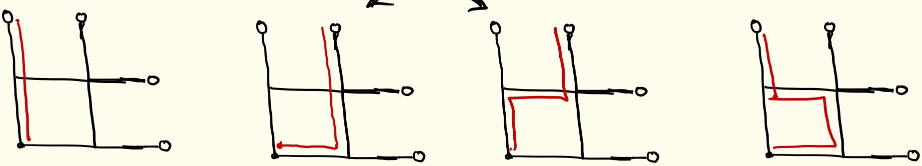
etc.

Let's consider mapping of SAWs that start at the origin and end at $2\hat{y}$ in the original setting, and therefore at $\hat{y}$ in the coarse-grained setting.

Denoting the original fugacity as K and renormalized fugacity as K',

$$K' = \underbrace{\sum}_{\substack{\text{sum over} \\ \text{all walks that} \\ \text{map to renormalized} \\ \text{walk}}} K^n \sum_R M_n(R)$$

$$= K^2 + 2K^3 + K^4$$



The equation $K' = K^4 + 2K^3 + K^2$ has three fixed points, $K^* = 0$, $K^* = \infty$ and $K^* \approx 0.46$. Based on the discussion above, the non-trivial fixed point where $\xi(K)$ diverges corresponds to $0 < K^* < 1$, therefore, $K_c \approx 0.46$. To find the critical exponent ν, we note that under coarse-graining, the number of SAW steps have halved, i.e. $\xi'(K') = \frac{\xi(K)}{b} = \frac{\xi(K)}{2}$, where $\xi(K)$ diverges as $\left[\frac{K_c - K}{K_c}\right]^{-\nu}$.

$$\Rightarrow \qquad \xi'(K') = \left[\frac{K_c - K'}{K_c}\right]^{-\nu} \qquad (i)$$

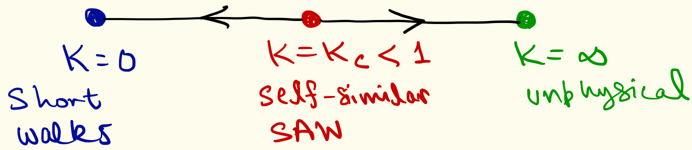
$$\text{and} \qquad \xi(K) = \left[\frac{K_c - K}{K_c}\right]^{-\nu} \qquad (ii)$$

dividing (i) by (ii) $\Rightarrow$ $\frac{1}{2} = \left[\frac{K_c - K'}{K_c - K}\right]^{-\nu}$

$$\Rightarrow \qquad \nu = \frac{\log 2}{\log\left[\left|\frac{K_c - K'}{K_c - K}\right|\right]}$$

Note that $\delta K' \equiv K_c - K' = \delta K \times 2^{1/\nu} > \delta K$

i.e. if one is close K_c, repeated RG transformations drive one away from the critical point. The RG flow in the K-space thus looks like:



Linearizing the RG eqn. $K' = K^4 + 2K^3 + 2K^2$

near K_c: $K' = K_c + \delta K'$, $K = K_c + \delta K$

$$\Rightarrow \quad K_c + \delta K' = (K_c + \delta K)^4 + 2(K_c + \delta K)^3 + 2(K_c + \delta K)^2$$

$$\Rightarrow \quad \delta K' \cong \delta K \left| \frac{\partial K'}{\partial K} \right|_{K_c}$$

$$\Rightarrow \quad \left| \frac{K' - K_c}{K - K_c} \right| = \frac{\partial K'}{\partial K}\Big|_{K_c}$$

$$= 4K_c^3 + 6K_c^2 + 4K_c$$

$$\Rightarrow \qquad \nu = \frac{\log 2}{\log\left[4K_c^3 + 6K_c^2 + 4K_c\right]}$$

$$\simeq \quad 0.72 \quad \text{using } K_c \simeq 0.46.$$

As mentioned earlier, $\nu_{exact} = \frac{3}{4} = 0.75$.

K_c depends on the lattice. for the current example, it's a square lattice and K_c is known to be approximately 0.38.